

Resource Theory of Special Antiunitary Asymmetry

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We propose the resource theory of a special antiunitary asymmetry in quantum theory. The notion of antiunitary asymmetry, in particular, \mathcal{PT} -asymmetry is different from the usual resource theory for asymmetry about unitary representation of a symmetry group, as the \mathcal{PT} operator is an antiunitary operator with \mathcal{P} being any self-inverse unitary and \mathcal{T} being the time-reversal operations. Here, we introduce the \mathcal{PT} -symmetric states, \mathcal{PT} -covariant operations and \mathcal{PT} -asymmetry measures. For single qubit system, we find duality relations between the \mathcal{PT} -asymmetry measures and the coherence. Moreover, for two-qubit states we prove the duality relations between the \mathcal{PT} -asymmetry measures and entanglement measure such as the concurrence. This gives a resource theoretic interpretation to the concurrence which is lacking till today. Thus, the \mathcal{PT} -asymmetry measure and entanglement can be viewed as two sides of an underlying resource. Finally, the \mathcal{PT} -symmetric dynamics is discussed and some open questions are addressed.

Introduction.— Quantum resource theories [1, 2] have played a pivotal role in the development and quantitative understanding of various physical phenomena in quantum physics and quantum information theory. A resource theory consists of two basic elements: free operations and free states. Any operation (or state) is dubbed as a resource if it falls out of the set of free operations (or the set of free states). The most significant resource theory is entanglement [3], which is a basic resource for various quantum information processing protocols, such as the superdense coding [4], teleportation [5] and remote state preparation [6]. The other notable examples include the resource theories of thermodynamics [7], asymmetry [8–14], coherence [15–22] and steering [23]. The main advantages of having a resource theory for some physical quantity are the succinct understanding of various physical processes and operational quantification of the relevant resources at ones disposal.

The Hermiticity is one fundamental requirement of quantum mechanics for the Hamiltonian of a quantum system, which guarantees that the energies are real and the total probability of the quantum state is conserved during the evolution of the system. However, it has been proved that a broad class of non-Hermitian Hamiltonian with \mathcal{PT} -symmetry can also have real spectra and probability conservation by redefined inner product [24–29], where \mathcal{P} denotes the parity operator and \mathcal{T} denotes the time reversal operator. This implies that \mathcal{PT} -symmetric theory constitutes a complex generalization of conventional quantum mechanics [26]. Moreover, in the system with \mathcal{PT} -symmetric non-Hermitian Hamiltonian, a number of interesting phenomena and applications appear in both classical and quantum regimes, such as unidirectional invisibility [30–32], non-Hermitian Bloch oscillation [33, 34], perfect

laser absorbers [35–37], ultrafast quantum state transformation [38], quantum state discrimination with single-shot measurement [39] and the potential violation of the no-signalling principle [40, 41]. However, most research focus on the \mathcal{PT} -symmetric Hamiltonian, never consider the quantum state with \mathcal{PT} -symmetry. Thus, the following questions arise: how to define a \mathcal{PT} -symmetric quantum state, what is the physical meaning of \mathcal{PT} -symmetric states and how to define measures of \mathcal{PT} -asymmetry.

Recently, the quantification of time reversal asymmetry [42] and \mathcal{CPT} asymmetry [43, 44] have been considered in antiunitary and unitary representations, respectively. However, there still remains a question in which representation to choose those relevant operations [45]. In this work, we use the framework of quantum resource theory to quantify \mathcal{PT} -asymmetry and investigate the relationship between \mathcal{PT} -asymmetry measures, quantum coherence and entanglement. Note that here \mathcal{P} is a self-inverse unitary operator (need not be parity operator) and \mathcal{T} is time-reversal operator. \mathcal{PT} operator can be realized as a special kind of antiunitary operator [46, 47], which is in contrast with the resource theory of asymmetry on the unitary representation of a symmetric group [8–14]. Thus, the resource of \mathcal{PT} -asymmetry will be a special kind of resource theory of antiunitary asymmetry. Though we cannot tensor the antilinear operator with the identity operator consistently, because antilinear operators are nonlocal, nevertheless they have been used to measure entanglement of a given bipartite state [48–53]. And there is a famous entanglement measure—the concurrence [49], which is indeed constructed from antilinear operators. It is quite satisfying that the resource theory of antiunitary asymmetry provides a unified view of two fundamental resources of quantum world such as the coherence and entanglement. For single qubit, we reveal a duality relation between the \mathcal{PT} -asymmetry measure and the coherence. For two-qubit pure states, we prove duality relations between the \mathcal{PT} -asymmetry measures and entanglement measure such as the concurrence. Amazingly, we find that the pure bipartite state is maximally entangled if and only

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if it is \mathcal{PT} -symmetric. Therefore, entanglement is a special \mathcal{PT} -symmetry in some sense. Furthermore, as $\mathcal{K} = *$ is an unphysical operator, that is \mathcal{K} cannot be realized in a physical system, then it is hard to calculate the \mathcal{PT} -asymmetry measure. However, we show that via the embedding quantum simulator [54–59], the \mathcal{PT} asymmetry measure can be calculated efficiently.

\mathcal{PT} -symmetric state. – Consider the self-inverse unitary operator \mathcal{P} and time reversal operator \mathcal{T} , where \mathcal{P} and \mathcal{T} satisfy the following condition: (1) $\mathcal{P} = \mathcal{P}^\dagger$, $\mathcal{P}^2 = I$, (2) $\mathcal{T} = U\mathcal{K}$, where U is a unitary operator with $U = U^t$ and $\mathcal{K} = *$ is the complex conjugation and (3) $[\mathcal{P}, \mathcal{T}] = 0$. Note that any antiunitary operator Θ with $\Theta = \Theta^\dagger = \Theta^{-1}$ can be written in the form $V\mathcal{K}$, where V is a unitary operator with $V = V^t$ and \mathcal{K} is the complex conjugation with respect to a given basis. Such antiunitary operator is called conjugation and plays an important role in quantum information theory [50, 60]. It is easy to see that such conjugation is equivalent to the \mathcal{PT} operator defined above. Thus, the resource theory of \mathcal{PT} -asymmetry considered in this work is a special kind of antiunitary asymmetry resource theory and may indicate the way towards formulating the general resource theory of antiunitary asymmetry.

Throughout this paper, we assume that self-inverse unitary operator and time reversal operator always satisfy these conditions. Given a quantum state ρ , once we apply the operations \mathcal{P} and \mathcal{T} , the final state will be $\mathcal{PT}\rho\mathcal{PT}$ (Since $\mathcal{PT}\rho\mathcal{PT} = \mathcal{P}U\mathcal{K}\rho\mathcal{K}U^*\mathcal{P} = \mathcal{P}U\rho^*U^\dagger\mathcal{P}$ is a quantum state). If the initial state is equal to the final state, that is $[\rho, \mathcal{PT}] = 0$, then we call the state ρ is \mathcal{PT} -symmetric state. If the initial state is not equal to the final state, that is $[\rho, \mathcal{PT}] \neq 0$, then we call the state ρ is \mathcal{PT} -asymmetric. Moreover, we denote the set of all \mathcal{PT} -symmetric states by $Sym(\mathcal{P}, \mathcal{T})$.

\mathcal{PT} -covariant operation. – To characterize the quantum operation which transform the \mathcal{PT} -symmetric states to the \mathcal{PT} -symmetric states, we distinguish quantum operations with and without subselection. Any quantum operation Φ can be described using a set of Kraus operators $\{K_\mu\}$ with $\Phi(\cdot) = \sum_\mu K_\mu(\cdot)K_\mu^\dagger$. The operation $\Phi : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$ is called \mathcal{PT} -covariant if $\Phi(\mathcal{PT}(\cdot)\mathcal{PT}) = \mathcal{PT}\Phi(\cdot)\mathcal{PT}$, that is $[\Phi, \mathcal{PT}] = 0$. Such operations are denoted by $\Phi_{\mathcal{PT}CO}$. (Of course, we can also consider the \mathcal{PT} -covariant operation with different \mathcal{PT} , that is $\Phi(\mathcal{P}_1\mathcal{T}_1(\cdot)\mathcal{P}_1\mathcal{T}_1) = \mathcal{P}_2\mathcal{T}_2\Phi(\cdot)\mathcal{P}_2\mathcal{T}_2$.) Besides this, we also need to consider the quantum operations with subselection. Thus, a quantum operation Φ is called selective \mathcal{PT} -covariant if the Kraus operators $\{K_\mu\}$ of Φ satisfy $K_\mu(\mathcal{PT}(\cdot)\mathcal{PT})K_\mu^\dagger = \mathcal{PT}K_\mu(\cdot)K_\mu^\dagger\mathcal{PT}$ for any μ .

The measure of \mathcal{PT} -asymmetry for a state. – When the state is \mathcal{PT} -asymmetric, that is it breaks the \mathcal{PT} -symmetry, we want to quantify how much the \mathcal{PT} -symmetry is broken by the given state. Thus, we need to introduce the \mathcal{PT} asymmetry measure, like the entanglement measure [61, 62], asymmetry measure [11, 14] and coherence measure [15, 16]. Now, we list the conditions that any function Γ from a state to a real number needs to satisfy in order to be a proper \mathcal{PT} -asymmetry measure.

For any proper \mathcal{PT} -asymmetry measure Γ , it needs to satisfy the following conditions:

(C1) $\Gamma(\rho, \mathcal{PT}) = 0$ iff $[\rho, \mathcal{PT}] = 0$.

(C2) Monotone under \mathcal{PT} -covariant operations $\Phi_{\mathcal{PT}CO}$, that is $\Gamma(\Phi_{\mathcal{PT}CO}(\rho), \mathcal{PT}) \leq \Gamma(\rho, \mathcal{PT})$.

(C2') Monotone under selective \mathcal{PT} -covariant operations: $\sum_\mu p_\mu \Gamma(\rho_\mu, \mathcal{PT}) \leq \Gamma(\rho, \mathcal{PT})$, where $K_\mu(\mathcal{PT}(\cdot)\mathcal{PT})K_\mu^\dagger = \mathcal{PT}K_\mu(\cdot)K_\mu^\dagger\mathcal{PT}$ and $\rho_\mu = K_\mu\rho K_\mu^\dagger/p_\mu$ with $p_\mu = \text{Tr}(K_\mu\rho K_\mu^\dagger)$.

(C3) Convexity: $\Gamma(\sum_n p_n \rho_n, \mathcal{PT}) \leq \sum_n p_n \Gamma(\rho_n, \mathcal{PT})$, where $\{\rho_n\}$ is a set of states and $p_n \geq 0$ with $\sum_n p_n = 1$.

The condition (C1) means the \mathcal{PT} -asymmetry measure vanishes if and only if this state is \mathcal{PT} -symmetric. We can weaken this condition as (C1'): $\Gamma(\rho, \mathcal{PT}) = 0$ if $[\rho, \mathcal{PT}] = 0$. Naturally, \mathcal{PT} -asymmetry measure cannot increase under the \mathcal{PT} -covariant operations, thus the conditions (C2) is necessary. Furthermore, the condition (C2') implies that the average \mathcal{PT} -asymmetry after the \mathcal{PT} -covariant operations with subselection cannot be greater than the \mathcal{PT} -asymmetry of the initial state. This condition may be important in real experiment so that we list this condition here. The condition (C3) is the convexity of the \mathcal{PT} -asymmetry measure, which is the requirement of any proper asymmetry monotone [11].

We now give several \mathcal{PT} -asymmetry measures via the relative entropy, the skew information and the fidelity measures.

Relative entropy of \mathcal{PT} -asymmetry. – The quantum relative entropy for states ρ and σ is defined as $S(\rho||\sigma) := \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma)$. The relative entropy of \mathcal{PT} -asymmetry measure Γ_r is defined as

$$\Gamma_r(\rho, \mathcal{PT}) = \min_{\sigma \in \text{Sym}(\mathcal{P}, \mathcal{T})} S(\rho||\sigma). \quad (1)$$

First, we get a closed form expression of Γ_r to avoid the minimization and it is given by

$$\Gamma_r(\rho, \mathcal{PT}) = S(\rho||\rho^{\mathcal{PT}}) = S(\rho^{\mathcal{PT}}) - S(\rho), \quad (2)$$

where $\rho^{\mathcal{PT}} = \frac{1}{2}(\rho + \mathcal{PT}\rho\mathcal{PT})$ is \mathcal{PT} -symmetric and Γ_r fulfills the conditions (C1), (C2), (C2') and (C3) as a proper \mathcal{PT} -asymmetry measure (See Appendix A). Then, for any state ρ , $\Gamma_r(\rho, \mathcal{PT}) = S(\rho^{\mathcal{PT}}) - S(\rho) \leq 1$, as $S(\sum_i p_i \rho_i) \leq \sum_i p_i S(\rho_i) + H(\{p_i\})$ [63] and $S(\mathcal{PT}\rho\mathcal{PT}) = S(\rho)$. Since $S(\sum_i p_i \rho_i) = \sum_i p_i S(\rho_i) + H(\{p_i\})$ is equivalent to that ρ_i have orthogonal supports [63], then $\Gamma_r(\rho, \mathcal{PT}) = 1$ iff $\rho \perp \mathcal{PT}\rho\mathcal{PT}$.

Skew information of \mathcal{PT} -asymmetry. – Let us define the skew information of \mathcal{PT} -asymmetry Γ_s as

$$\begin{aligned} \Gamma_s(\rho, \mathcal{PT}) &:= -\frac{1}{2} \text{Tr} \left([\rho^{1/2}, \mathcal{PT}]^2 \right) \\ &= 1 - \text{Tr} \left(\rho^{1/2} \mathcal{PT} \rho^{1/2} \mathcal{PT} \right), \end{aligned} \quad (3)$$

where $[\cdot, \cdot]$ denote the commutator and $[\rho^{1/2}, \mathcal{PT}]^2 = \rho + \mathcal{PT}\rho\mathcal{PT} - 2\rho^{1/2}\mathcal{PT}\rho^{1/2}\mathcal{PT}$. Note that, in the definition of Wigner-Yanase-Dyson skew information $I(\rho, O) = -\frac{1}{2} \text{Tr}([\rho^{1/2}, O])$, the operator O is required to be an observable [64], that is O must be a Hermitian, however \mathcal{PT} is not a linear operator, thus \mathcal{PT} is not an observable. Therefore, we cannot use the properties of skew information to state that

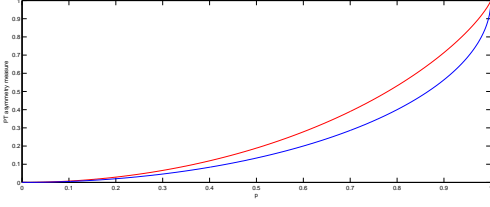


FIG. 1. The plot shows the \mathcal{PT} -asymmetry measure Γ_r (red line) and Γ_s (blue line) for qubit states $\rho = (1-p)\mathbb{I}/2 + p|\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $p \in [0, 1]$ under the unitary operator $\mathcal{P} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and time reversal operator $\mathcal{T} = *$

Γ_s satisfy the conditions (C1), (C2), (C2') and (C3). However, Γ_s still fulfills these conditions (See Appendix B). Obviously, for any state ρ , $\Gamma_s(\rho, \mathcal{PT}) \leq 1$ and the equality holds iff $\rho \perp \mathcal{PT}\rho\mathcal{PT}$.

The above two quantities Γ_r and Γ_s are proper \mathcal{PT} -asymmetry measures (An example is presented in Fig.1). Of course, there may be other possible \mathcal{PT} -asymmetry measure, like the \mathcal{PT} -asymmetry measure induced by the trace norm, the Hilbert Schmidt norm and so on. Here, we introduce another interesting \mathcal{PT} -asymmetry measure defined by the fidelity.

Fidelity measure of \mathcal{PT} -asymmetry. – Let us consider the \mathcal{PT} -asymmetry measure defined as

$$\begin{aligned} \Gamma_F(\rho, \mathcal{PT}) &= 1 - F(\rho, \mathcal{PT}\rho\mathcal{PT}) \\ &= 1 - \text{Tr} \left(\sqrt{\sqrt{\rho}\mathcal{PT}\rho\mathcal{PT}\sqrt{\rho}} \right), \end{aligned} \quad (4)$$

which fulfils the conditions (C1), (C2), (C2') and (C3) (See Appendix C).

Based on the proof in Proposition 7 (See Appendix B), we have $\Gamma_F \leq \Gamma_s$. Following Ref.[50], $\Gamma_F(\rho, \mathcal{PT})$ can be written as

$$\Gamma_F(\rho, \mathcal{PT}) = \min_k \sum_k p_k \Gamma_F(\psi_k, \mathcal{PT}), \quad (5)$$

where the minimum is taken over all the decomposition of $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$. Furthermore, the optimal decomposition can be found in Ref.[50].

Duality of \mathcal{PT} -Asymmetry, Coherence and Entanglement.–

Given a self-inverse unitary operator \mathcal{P} and a time reversal operator \mathcal{T} , for any pure state $\rho = |\psi\rangle\langle\psi|$, we have

$$\Gamma_s(\psi, \mathcal{PT}) = 1 - |\langle\psi|\mathcal{PT}|\psi\rangle|^2, \quad (6)$$

$$\text{and } \Gamma_F(\psi, \mathcal{PT}) = 1 - |\langle\psi|\mathcal{PT}|\psi\rangle|. \quad (7)$$

This can be interpreted as follows. Imagine that we have two copies of a pure state $|\psi\rangle$, and one is rotated in space by a unitary operator \mathcal{P} and the other is transformed under time reversal operator \mathcal{T} . The final states will be $\mathcal{P}|\psi\rangle$ and $\mathcal{T}|\psi\rangle$, and we want to know whether these final states coincide or not. If they coincide, then this means that the effect of the parity operator and the time reversal operator leaves the state

$|\psi\rangle$ invariant, and we say $|\psi\rangle$ has \mathcal{PT} -symmetry. Otherwise, the state $|\psi\rangle$ breaks the \mathcal{PT} -symmetry.

Moreover, the spectrum of $\rho^{\mathcal{PT}}$ with $\rho = |\psi\rangle\langle\psi|$ is $\{\frac{1}{2} - \frac{1}{2}|\langle\psi|\mathcal{PT}|\psi\rangle|, \frac{1}{2} + \frac{1}{2}|\langle\psi|\mathcal{PT}|\psi\rangle|\}$. Thus,

$$\Gamma_r(\psi, \mathcal{PT}) = H\left(\frac{1}{2} - \frac{1}{2}|\langle\psi|\mathcal{PT}|\psi\rangle|\right) \quad (8)$$

where $H(p) = -p\log(p) - (1-p)\log(1-p)$ is the Shannon entropy for the probability distribution $\{p, 1-p\}$.

Let us consider the simplest case: a single qubit system. We take $\mathcal{P} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{T} = *$. Then for pure qubit state $|\psi\rangle = (\psi_1, \psi_2)^t$, where t denotes the transpose,

$$\begin{aligned} \Gamma_s(\psi, \mathcal{PT}) &= 1 - |\langle\psi|\mathcal{PT}|\psi\rangle|^2 \\ &= 1 - |\psi_1\psi_2 + \psi_2\psi_1|^2 \\ &= 1 - 4|\psi_1|^2|\psi_2|^2, \end{aligned}$$

and

$$\Gamma_r(\psi, \mathcal{PT}) = H\left(\frac{1}{2} - 2|\psi_1|^2|\psi_2|^2\right).$$

Thus, a pure state ψ is \mathcal{PT} -symmetric iff $|\psi_1| = |\psi_2| = 1/\sqrt{2}$. This suggests that there should be a connection between the quantum coherence [15] and the \mathcal{PT} -asymmetry measure. In fact, for a single qubit system we have duality relations between the l_1 -norm of coherence and the \mathcal{PT} -asymmetry measures as given by

$$\begin{aligned} \Gamma_s(\psi, \mathcal{PT}) + C_{l_1}(\psi)^2 &= 1, \\ \Gamma_F(\psi, \mathcal{PT}) + C_{l_1}(\psi) &= 1, \end{aligned} \quad (9)$$

where $C_{l_1}(\psi) = \sum_{i \neq j} |\rho_{ij}| = 2|\psi_1||\psi_2|$ is the l_1 -norm of coherence for a single qubit. Therefore, a maximally pure coherent state is actually a \mathcal{PT} -symmetric state.

However, in two-qubit system, we have two different ways to consider the \mathcal{PT} -asymmetry. On the one hand, we can construct the \mathcal{P}, \mathcal{T} operators on 2-qubit system using \mathcal{P}, \mathcal{T} operators on single qubit systems like $\mathcal{P}_1\mathcal{T}_1 \otimes \mathcal{P}_2\mathcal{T}_2$. On the other hand, we can construct \mathcal{P}, \mathcal{T} operators on 2-qubit system which cannot be constructed from single qubit systems, and this may be connected with entanglement closely.

For two-qubit pure state $|\Psi\rangle$ the famous entanglement monotone—the concurrence [49] is defined as

$$C(\Psi) = |\langle\Psi|\sigma_y \otimes \sigma_y \mathcal{K}|\Psi\rangle|. \quad (10)$$

Now, we prove duality relations between the \mathcal{PT} -asymmetry measures and the concurrence. Using the definitions of Γ_s, Γ_r and $C(\psi)$ for any pure two qubit state, we have the following theorem.

Theorem 1. *Given a two-qubit system with self-inverse unitary operator $\mathcal{P} = \sigma_y \otimes \sigma_y$ and time reversal operator $\mathcal{T} = *$, for pure bipartite states $|\Psi\rangle$ we have*

$$\Gamma_s(\Psi, \mathcal{PT}) + C(\Psi)^2 = 1, \quad (11)$$

$$\Gamma_F(\Psi, \mathcal{PT}) + C(\Psi) = 1, \quad (12)$$

and

$$\Gamma_r(\Psi, \mathcal{PT}) = H\left(\frac{1}{2} - \frac{1}{2}C(\Psi)\right), \quad (13)$$

where $H(p) = -p \log(p) - (1-p) \log(1-p)$ is the Shannon entropy for the probability distribution $\{p, 1-p\}$ and $C(\Psi)$ is the concurrence for pure state $|\Psi\rangle$.

For any two-qubit mixed states ρ , the equalities may not hold. However, we still have the following inequality:

$$\Gamma_s(\rho, \mathcal{PT}) + C(\rho)^2 \leq 1, \quad (14)$$

$$\Gamma_F(\rho, \mathcal{PT}) + C(\rho) \leq 1, \quad (15)$$

$$\Gamma_r(\rho, \mathcal{PT}) \leq H\left(\frac{1}{2} - \frac{1}{2}C(\rho)\right), \quad (16)$$

where $C(\rho) = \min \sum_k p_k C(\Psi_k)$ and the minimum is taken over all the pure states decomposition of $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ [49, 50].

In fact, we can prove the following

$$\Gamma_F(\rho, \mathcal{PT}) + CoA(\rho) = 1, \quad (17)$$

where the concurrence of assistance $CoA(\rho) = \max \sum_k p_k C(\Psi_k)$ and the maximum is taken over all the pure states decomposition of $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ [65, 66].

The proof of this theorem is presented in the Appendix D.

Since the concurrence quantifies the entanglement of a pure bipartite state, thus the above proposition shows that a pure state has more \mathcal{PT} -symmetry with $\mathcal{P} = \sigma_y \otimes \sigma_y$ and $\mathcal{T} = *$ if and only if this state is more entangled, i.e., the pure state Ψ is a \mathcal{PT} -symmetric state iff Ψ is maximally entangled. Therefore, entanglement is a special kind of \mathcal{PT} -symmetry in some sense. Our formalism, the resource theory of antiunitary asymmetry, in fact, provides a unified view of two fundamental resources such as the quantum coherence and the entanglement.

To generalize these notions, we consider the relationship between \mathcal{PT} -asymmetry and entanglement in multi-qubit system. In N -qubit system with $N \geq 2$, there exists a systematic procedure to define entanglement monotone for pure states via three operational building blocks [52, 55]: $\mathcal{K} = *$, σ_y and $g_{ij}\sigma_i\sigma_j$, where $g_{ij} = \text{diag}\{-1, 1, 0, 1\}$, $\sigma_0 = I$, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$ and $\sigma_3 = \sigma_z$. If N is even, then the simplest entanglement monotone is $|\langle\Psi|\sigma_y^{\otimes N}\mathcal{K}|\Psi\rangle|$ and if N is odd, $|\sum_{ij} g_{ij} \langle\Psi|\sigma_i \otimes \sigma_j^{\otimes N-1}\mathcal{K}|\Psi\rangle \langle\Psi|\sigma_j \otimes \sigma_i^{\otimes N-1}\mathcal{K}|\Psi\rangle|$ is the simplest entanglement monotone [52, 55]. With this construction, for $N=2$, we have the entanglement monotone—the concurrence: $C(\Psi) = |\langle\Psi|\sigma_y \otimes \sigma_y \mathcal{K}|\Psi\rangle|$. For $N=3$, we have the 3-tangle [51] defined as $\tau_3(\Psi) := |\sum_{ij} g_{ij} \langle\Psi|\sigma_i \otimes \sigma_j^{\otimes 2} \mathcal{K}|\Psi\rangle \langle\Psi|\sigma_j \otimes \sigma_i^{\otimes 2} \mathcal{K}|\Psi\rangle|$.

Thus, we have defined entanglement monotone τ_N for any N -qubit system and it is easy to see that for the even and odd cases we have the following relations:

(i) if $N = 2k$, then

$$\Gamma_s(\Psi, \mathcal{P}_2\mathcal{T}) + \tau_N(\Psi)^2 = 1, \quad (18)$$

$$\Gamma_r(\Psi, \mathcal{P}_2\mathcal{T}) = H\left(\frac{1}{2} - \frac{1}{2}\tau_N(\Psi)\right), \quad (19)$$

where $\mathcal{P}_2 = \sigma_y^{\otimes N}$, $\mathcal{T} = *$ and $H(p) = -p \log(p) - (1-p) \log(1-p)$ is the Shannon entropy for the probability distribution $\{p, 1-p\}$.

(ii) if $N = 2k + 1$, then

$$\tau_N(\Psi) = |-\Gamma_s(\Psi, \mathcal{P}_0\mathcal{T}) + \Gamma_s(\Psi, \mathcal{P}_2\mathcal{T}) + \Gamma_s(\Psi, \mathcal{P}_3\mathcal{T}) - 1|, \quad (20)$$

where $\mathcal{P}_i = \sigma_i \otimes \sigma_y^{\otimes N-1}$ and $\mathcal{T} = *$.

Since $\mathcal{K} = *$ is an unphysical operator, one may think that we need to perform full tomography to calculate $\langle\Psi|\mathcal{PT}|\Psi\rangle = \langle\Psi|\mathcal{P}\mathcal{U}\mathcal{K}|\Psi\rangle$ [54, 55]. However, based on the embedding quantum simulator (EQS), such quantity can be calculated efficiently [54, 55]. This technique of embedding quantum simulator [54–59] is described as follows: define a mapping $\mathcal{M} : \mathbb{C}^d \rightarrow \mathbb{R}^{2d}$ as

$$|\Psi\rangle = \begin{pmatrix} \psi_{re}^1 \\ \psi_{re}^2 + i\psi_{im}^1 \\ \psi_{re}^3 + i\psi_{im}^2 \\ \vdots \end{pmatrix} \rightarrow |\tilde{\Psi}\rangle = \begin{pmatrix} \psi_{re}^1 \\ \psi_{re}^2 \\ \psi_{re}^3 \\ \vdots \\ \psi_{im}^1 \\ \psi_{im}^2 \\ \psi_{im}^3 \\ \vdots \end{pmatrix} \quad (21)$$

The reverse mapping is given by $|\Psi\rangle = M|\tilde{\Psi}\rangle$, with $M = (1, i) \otimes I_d$ and $\mathcal{K}|\Psi\rangle = M(\sigma_z \otimes I_d)|\tilde{\Psi}\rangle$. Thus, one has

$$\langle\Psi|\mathcal{P}\mathcal{U}\mathcal{K}|\Psi\rangle = \langle\tilde{\Psi}|M^\dagger\mathcal{P}\mathcal{U}M(\sigma_z \otimes I_d)|\tilde{\Psi}\rangle. \quad (22)$$

By the embedding quantum simulator, the quantity $|\langle\Psi|\mathcal{PT}|\Psi\rangle|$ can be calculated efficiently, which means the \mathcal{PT} -asymmetry measures such as $\Gamma_r, \Gamma_s, \Gamma_F$ can be calculated efficiently (see Fig. 2).

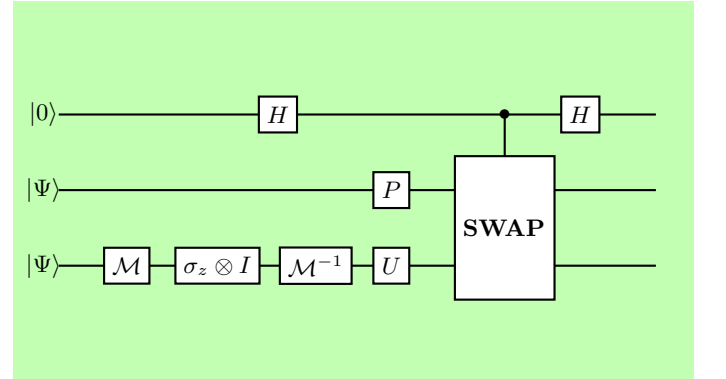


FIG. 2. A quantum network for estimation of \mathcal{PT} -asymmetry. The probability of finding the control qubit (the top line) in state $|0\rangle$: p_0 depends on the \mathcal{PT} asymmetry of $|\Psi\rangle$ [67, 68], that is $p_0 = (1 + |\langle\Psi|\mathcal{PT}|\Psi\rangle|^2)/2$, where $\mathcal{T} = \mathcal{U}\mathcal{K}$.

Conclusion.— To summarize, here we have developed the resource theory of a special kind of antiunitary asymmetry, namely, the \mathcal{PT} -asymmetry with \mathcal{P} being any self-inverse unitary and \mathcal{T} being the time-reversal. We have introduced

the notion of \mathcal{PT} -symmetric states, \mathcal{PT} -covariant operations and \mathcal{PT} -asymmetry measures. We give several interesting \mathcal{PT} -asymmetry measures which are induced from different distance measures. Most importantly, we have proved the duality relations between the \mathcal{PT} -asymmetry measures, coherence and concurrence. This also gives new interpretations to quantum coherence and entanglement, which are special \mathcal{PT} -symmetries in some sense, thus unifying two fundamental resources of quantum world. Furthermore, we have argued that via the embedding quantum simulator, the \mathcal{PT} -asymmetry measures can be calculated efficiently. Finally, we have discussed the \mathcal{PT} -symmetric dynamics and proposed several open problems. Our findings will open up new ways of thinking about quantum coherence and entanglement from an-

other resource theoretic point of view, i.e., the \mathcal{PT} -asymmetry measures. The \mathcal{PT} -asymmetry is a just special kind of antiunitary asymmetry resource theory and may pave the way to a general antiunitary asymmetry resource theory.

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- [1] M. Horodecki and J. Oppenheim, *Int. J. Mod. Phys. B* **27**, 1345019 (2013).
 - [2] F. G. S. L. Brandão and G. Gour, *Phys. Rev. Lett.* **115**, 070503 (2015).
 - [3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
 - [4] C. H. Bennett and S. J. Wiesner, *Phys. Rev. Lett.* **69**, 2881 (1992).
 - [5] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).
 - [6] A. K. Pati, *Phys. Rev. A* **63**, 014302 (2000).
 - [7] F. G. S. L. Brandão, M. Horodecki, J. Oppenheim, J. M. Renes, and R. W. Spekkens, *Phys. Rev. Lett.* **111**, 250404 (2013).
 - [8] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, *Rev. Mod. Phys.* **79**, 555 (2007).
 - [9] G. Gour and R. W. Spekkens, *New J. Phys.* **10**, 033023 (2008).
 - [10] G. Gour, I. Marvian, and R. W. Spekkens, *Phys. Rev. A* **80**, 012307 (2009).
 - [11] I. Marvian Mashhad, *Symmetry, Asymmetry and Quantum Information* (PhD thesis, University of Waterloo, 2012).
 - [12] I. Marvian and R. W. Spekkens, *New J. Phys.* **15**, 033001 (2013).
 - [13] I. Marvian and R. W. Spekkens, *Phys. Rev. A* **90**, 062110 (2014).
 - [14] I. Marvian and R. W. Spekkens, *Nat. Commun.* **5**, 3821 (2014).
 - [15] T. Baumgratz, M. Cramer, and M. B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
 - [16] D. Girolami, *Phys. Rev. Lett.* **113**, 170401 (2014).
 - [17] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, *Phys. Rev. Lett.* **115**, 020403 (2015).
 - [18] A. Winter and D. Yang, *Phys. Rev. Lett.* **116**, 120404 (2016).
 - [19] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, *Phys. Rev. Lett.* **116**, 080402 (2016).
 - [20] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, *Phys. Rev. Lett.* **116**, 150502 (2016).
 - [21] E. Chitambar, A. Streltsov, S. Rana, M. N. Bera, G. Adesso, and M. Lewenstein, *Phys. Rev. Lett.* **116**, 070402 (2016).
 - [22] E. Chitambar and G. Gour, *Phys. Rev. Lett.* **117**, 030401 (2016).
 - [23] R. Gallego and L. Aolita, *Phys. Rev. X* **5**, 041008 (2015).
 - [24] C. M. Bender and S. Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998).
 - [25] C. M. Bender, S. Boettcher, and P. N. Meisinger, *J. Math. Phys.* **40**, 2201 (1999).
 - [26] C. M. Bender, D. C. Brody, and H. F. Jones, *Phys. Rev. Lett.* **89**, 270401 (2002).
 - [27] C. M. Bender, J. Brod, A. Refig, and M. E. Reuter, *J. Phys. A: Math. Gen.* **37**, 10139 (2004).
 - [28] C. M. Bender, J.-H. Chen, and K. A. Milton, *J. Phys. A: Math. Gen.* **39**, 1657 (2006).
 - [29] C. M. Bender, *Rep. Prog. Phys.* **70**, 947 (2007).
 - [30] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, *Phys. Rev. Lett.* **106**, 213901 (2011).
 - [31] L. Feng, Y.-L. Xu, W. S. Fegadolli, M.-H. Lu, J. B. Oliveira, V. R. Almeida, Y.-F. Chen, and A. Scherer, *Nat. Mater.* **12**, 108 (2013).
 - [32] X. Yin and X. Zhang, *Nat. Mater.* **12**, 175 (2013).
 - [33] S. Longhi, *Phys. Rev. Lett.* **103**, 123601 (2009).
 - [34] M. Wimmer, M.-A. Miri, D. Christodoulides, and U. Peschel, *Sci. Rep.* **5**, 17760 (2015).
 - [35] Y. D. Chong, L. Ge, H. Cao, and A. D. Stone, *Phys. Rev. Lett.* **105**, 053901 (2010).
 - [36] S. Longhi, *Phys. Rev. A* **82**, 031801 (2010).
 - [37] S. Longhi and L. Feng, *Opt. Lett.* **39**, 5026 (2014).
 - [38] C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, *Phys. Rev. Lett.* **98**, 040403 (2007).
 - [39] C. M. Bender, D. C. Brody, J. Caldeira, U. Günther, B. K. Meister, and B. F. Samsonov, *Phil. Trans. R. Soc. A* **371** (2013), 10.1098/rsta.2012.0160.
 - [40] Y.-C. Lee, M.-H. Hsieh, S. T. Flammia, and R.-K. Lee, *Phys. Rev. Lett.* **112**, 130404 (2014).
 - [41] J.-S. Tang, Y.-T. Wang, S. Yu, D.-Y. He, J.-S. Xu, B.-H. Liu, G. Chen, Y.-N. Sun, K. Sun, Y.-J. Han, C.-F. Li, and G.-C. Guo, *Nat. photon.* (2016).
 - [42] G. Gour, B. C. Sanders, and P. S. Turner, *J. Math. Phys.* **50**, 102105 (2009).
 - [43] M. Skotiniotis, B. Toloui, I. T. Durham, and B. C. Sanders, *Phys. Rev. Lett.* **111**, 020504 (2013).
 - [44] M. Skotiniotis, B. Toloui, I. T. Durham, and B. C. Sanders, *Phys. Rev. A* **90**, 012326 (2014).
 - [45] P. Kosiński, *Phys. Rev. Lett.* **114**, 058902 (2015).
 - [46] C. E. Porter, *Statistical theories of spectra: fluctuations* (1965).
 - [47] C. M. Bender, M. V. Berry, and A. Mandilara, *J. Phys. A: Math. Gen.* **35**, L467 (2002).
 - [48] S. Hill and W. K. Wootters, *Phys. Rev. Lett.* **78**, 5022 (1997).
 - [49] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
 - [50] A. Uhlmann, *Phys. Rev. A* **62**, 032307 (2000).
 - [51] V. Coffman, J. Kundu, and W. K. Wootters, *Phys. Rev. A* **61**, 052306 (2000).

- [52] A. Osterloh and J. Siewert, *Phys. Rev. A* **72**, 012337 (2005).
- [53] T. J. Osborne and F. Verstraete, *Phys. Rev. Lett.* **96**, 220503 (2006).
- [54] J. Casanova, C. Sabín, J. León, I. L. Egusquiza, R. Gerritsma, C. F. Roos, J. J. García-Ripoll, and E. Solano, *Phys. Rev. X* **1**, 021018 (2011).
- [55] R. Di Candia, B. Mejia, H. Castillo, J. S. Pedernales, J. Casanova, and E. Solano, *Phys. Rev. Lett.* **111**, 240502 (2013).
- [56] I. M. Georgescu, S. Ashhab, and F. Nori, *Rev. Mod. Phys.* **86**, 153 (2014).
- [57] X. Zhang, Y. Shen, J. Zhang, J. Casanova, L. Lamata, E. Solano, M.-H. Yung, J.-N. Zhang, and K. Kim, *Nat. Commun.* **6**, 7917 (2015).
- [58] J. C. Loredó, M. P. Almeida, R. Di Candia, J. S. Pedernales, J. Casanova, E. Solano, and A. G. White, *Phys. Rev. Lett.* **116**, 070503 (2016).
- [59] M.-C. Chen, D. Wu, Z.-E. Su, X.-D. Cai, X.-L. Wang, T. Yang, L. Li, N.-L. Liu, C.-Y. Lu, and J.-W. Pan, *Phys. Rev. Lett.* **116**, 070502 (2016).
- [60] A. Uhlmann, *Sci. China Phys. Mech. Astron.* **59**, 1 (2016).
- [61] V. Vedral and M. B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).
- [62] M. B. Plenio and S. Virmani, *Quantum Info. Comput.* **7**, 001 (2007).
- [63] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, 2010).
- [64] E. P. Wigner and M. M. Yanase, *Proc. Natl. Acad. Sci. U.S.A.* **49**, 910 (1963).
- [65] T. Laustsen, F. Verstraete, and S. J. Van Enk, *Quantum Info. Comput.* **3**, 64 (2003).
- [66] G. Gour, D. A. Meyer, and B. C. Sanders, *Phys. Rev. A* **72**, 042329 (2005).
- [67] A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, *Phys. Rev. Lett.* **88**, 217901 (2002).
- [68] E. Sjöqvist, A. K. Pati, A. Ekert, J. S. Anandan, M. Ericsson, D. K. L. Oi, and V. Vedral, *Phys. Rev. Lett.* **85**, 2845 (2000).
- [69] G. Lindblad, *Commun. Math. Phys.* **40**, 147 (1975).
- [70] M. B. Ruskai, *J. Math. Phys.* **43** (2002).
- [71] M. Fannes, *Commun. Math. Phys.* **31**, 291 (1973).
- [72] E. H. Lieb, *Advan. Math.* **11**, 267 (1973).
- [73] M. Hayashi, *Quantum Information: An introduction* (Springer, 2006).
- [74] J. A. Miszczak, Z. Puchała, P. Horodecki, A. Uhlmann, and K. Życzkowski, *Quantum Info. Comput.* **9**, 103 (2009).
- [75] M. Fannes, F. De Melo, W. Roga, and K. Życzkowski, *Quantum Info. Comput.* **12**, 472 (2012).
- [76] C. A. Fuchs and J. van de Graaf, *IEEE Trans. Inf. Theory* **45**, 1216 (1999).
- [77] M. A. Nielsen, *Phys. Rev. Lett.* **83**, 436 (1999).
- [78] S. Du, Z. Bai, and Y. Guo, *Phys. Rev. A* **91**, 052120 (2015).

Appendix A: Relative entropy of \mathcal{PT} -asymmetry

To prove the properties of Γ_s , we need the following lemma, which is not trivial, as \mathcal{PT} is antilinear operator.

Lemma 2. *Given the self-inverse unitary operator \mathcal{P} and time reversal operator \mathcal{T} , for any two Hermitian operators Q and S , we have*

$$\text{Tr}(Q\mathcal{P}\mathcal{T}S\mathcal{P}\mathcal{T}) = \text{Tr}(\mathcal{P}\mathcal{T}Q\mathcal{P}\mathcal{T}S). \quad (\text{A1})$$

Proof. Due to the spectral decomposition Theorem, Q and S can be written as $Q = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ and $S = \sum_j \mu_j |\phi_j\rangle\langle\phi_j|$, respectively. Thus, we have

$$\begin{aligned} & \text{Tr}(Q\mathcal{P}\mathcal{T}S\mathcal{P}\mathcal{T}) \\ &= \sum_{ij} \lambda_i \mu_j \text{Tr}(|\psi_i\rangle\langle\psi_i| \mathcal{P}U\mathcal{K}|\phi_j\rangle\langle\phi_j| \mathcal{K}U^\dagger \mathcal{P}) \\ &= \sum_{ij} \lambda_i \mu_j |\langle\psi_i| \mathcal{P}U |\phi_j^*\rangle|^2, \end{aligned}$$

where the first equality comes from the fact that $\mathcal{T} = U\mathcal{K}$, where U is a unitary operator with $U = U^\dagger$ and $\mathcal{K} = *$. Similarly, we have

$$\text{Tr}(\mathcal{P}\mathcal{T}Q\mathcal{P}\mathcal{T}S) = \sum_{ij} \lambda_i \mu_j |\langle\phi_j| \mathcal{P}U |\psi_i^*\rangle|^2.$$

Hence, to prove $\text{Tr}(Q\mathcal{P}\mathcal{T}S\mathcal{P}\mathcal{T}) = \text{Tr}(\mathcal{P}\mathcal{T}Q\mathcal{P}\mathcal{T}S)$, we only need to prove that for any two pure states $|\psi\rangle$ and $|\phi\rangle$

$$|\langle\psi| V |\phi^*\rangle| = |\langle\phi| V |\psi^*\rangle|, \quad (\text{A2})$$

where $V = \mathcal{P}U$ is a unitary operator. Moreover, since $\mathcal{P} = \mathcal{T}\mathcal{P}\mathcal{T} = U\mathcal{K}\mathcal{P}U\mathcal{K} = U\mathcal{P}^*U^*$, then $\mathcal{P}^*U^* = U^\dagger\mathcal{P} = U^\dagger\mathcal{P}^\dagger$, which implies that $U^\dagger\mathcal{P}^\dagger = \mathcal{P}U$. That is, $V^\dagger = V$. Therefore, $|\langle\psi| V |\phi^*\rangle| = |\langle\phi^*| V^\dagger |\psi\rangle| = |\langle\phi| V |\psi^*\rangle|$ \square

Proposition 3. *Given the self-inverse unitary operator \mathcal{P} and time reversal operator \mathcal{T} , let $\Gamma_r(\rho, \mathcal{P}\mathcal{T}) = \min_{\sigma \in \text{Sym}(\mathcal{P}, \mathcal{T})} S(\rho||\sigma)$, then we have*

$$\Gamma_r(\rho, \mathcal{P}\mathcal{T}) = S(\rho||\rho^{\mathcal{P}\mathcal{T}}) = S(\rho^{\mathcal{P}\mathcal{T}}) - S(\rho), \quad (\text{A3})$$

where $\rho^{\mathcal{P}\mathcal{T}} = \frac{1}{2}(\rho + \mathcal{P}\mathcal{T}\rho\mathcal{P}\mathcal{T})$ is \mathcal{PT} -symmetric.

Proof. Since it involves the complex conjugation \mathcal{K} , taking trace may be complicated. As for any linear operator A , $\mathcal{K}A\mathcal{K} = A^*$ and $\text{Tr}(A^*) \neq \text{Tr}(A)$ in general, thus we need be more careful to deal with taking trace here. However, due to Lemma 2, for any two Hermitian operators Q and S , we have

$$\text{Tr}(Q\mathcal{P}\mathcal{T}S\mathcal{P}\mathcal{T}) = \text{Tr}(\mathcal{P}\mathcal{T}Q\mathcal{P}\mathcal{T}S). \quad (\text{A4})$$

Then we follow the approach in [10, 11] to complete the proof. Due to the fact that $S(\rho||\sigma) \geq 0$ and $S(\rho||\sigma) = 0$ iff $\sigma = \rho$, thus

$$\text{Tr}(\rho \log \sigma) \leq \text{Tr}(\rho \log \rho) \quad (\text{A5})$$

where the equality holds iff $\sigma = \rho$, which implies that $\max_{\sigma} \text{Tr}(\rho \log \sigma) = \text{Tr}(\rho \log \rho)$.

First, for any \mathcal{PT} -symmetric state σ , we have

$$\text{Tr}(\mathcal{P}\mathcal{T}\rho\mathcal{P}\mathcal{T} \log \sigma) = \text{Tr}(\rho\mathcal{P}\mathcal{T} \log \sigma\mathcal{P}\mathcal{T}) = \text{Tr}(\rho \log \sigma),$$

where the first equality comes from the equation (A4) and the second equality comes from the fact that σ is \mathcal{PT} -symmetric. Hence, we have

$$\begin{aligned} & \text{Tr}(\rho \log \sigma) \\ &= \frac{1}{2} \text{Tr}(\rho \log \sigma) + \frac{1}{2} \text{Tr}(\mathcal{P}\mathcal{T}\rho\mathcal{P}\mathcal{T} \log \sigma) \\ &= \text{Tr}(\rho^{\mathcal{P}\mathcal{T}} \log \sigma), \end{aligned} \quad (\text{A6})$$

where $\rho^{\mathcal{PT}} = \frac{1}{2}(\rho + \mathcal{PT}\rho\mathcal{PT})$. Thus, for the \mathcal{PT} -symmetric state $\rho^{\mathcal{PT}}$, $S(\rho||\rho^{\mathcal{PT}}) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \rho^{\mathcal{PT}}) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho^{\mathcal{PT}} \log \rho^{\mathcal{PT}}) = S(\rho^{\mathcal{PT}}) - S(\rho)$.

Next, we show that $\min_{\sigma \in \text{Sym}(\mathcal{P}, \mathcal{T})} S(\rho||\sigma) = S(\rho||\rho^{\mathcal{PT}})$.

$$\begin{aligned} & \min_{\sigma \in \text{Sym}(\mathcal{P}, \mathcal{T})} S(\rho||\sigma) \\ &= \min_{\sigma \in \text{Sym}(\mathcal{P}, \mathcal{T})} [\text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma)] \\ &= \text{Tr}(\rho \log \rho) - \max_{\sigma \in \text{Sym}(\mathcal{P}, \mathcal{T})} \text{Tr}(\rho \log \sigma) \\ &= \text{Tr}(\rho \log \rho) - \max_{\sigma \in \text{Sym}(\mathcal{P}, \mathcal{T})} \text{Tr}(\rho^{\mathcal{PT}} \log \sigma) \\ &= \text{Tr}(\rho \log \rho) - \text{Tr}(\rho^{\mathcal{PT}} \log \rho^{\mathcal{PT}}) \\ &= S(\rho^{\mathcal{PT}}) - S(\rho), \end{aligned}$$

where the third equality comes from (A6) and the fourth equality comes from the inequality (A5). Therefore, the proof of the proposition is completed. \square

Proposition 4. *Given the self-inverse unitary operator \mathcal{P} and time reversal operator \mathcal{T} , then Γ_r satisfy the conditions (C1), (C2), (C2') and (C3), so it is a proper \mathcal{PT} -asymmetry measure.*

Proof. Since $S(\rho||\sigma) = 0$ iff $\rho = \sigma$, thus Γ_r satisfy (C1). Besides, as the relative entropy is contracted under quantum operations [69] and jointly convex [70], then Γ_r satisfy the conditions (C2) and (C3). Moreover, we use the techniques in Ref.[11, 15] to prove that Γ_r satisfy the condition (C2').

Take a special self-inverse unitary operator $\mathcal{P}_0 = I$ and a time reversal operator $\mathcal{T}_0 = *$, then it is easy to see that there exist a set of orthonormal pure states $\{|\mu\rangle\}_\mu$ with $|\mu\rangle\langle\mu| \in \text{Sym}(\mathcal{P}_0, \mathcal{T}_0)$. For any selective \mathcal{PT} -covariant operation Φ with $K_\mu(\mathcal{PT}(\cdot)\mathcal{PT})K_\mu^\dagger = \mathcal{PT}K_\mu(\cdot)K_\mu^\dagger\mathcal{PT}$ for any μ , it is easy to verify that the quantum operations $\tilde{\Phi}$ with Kraus operators $\tilde{K}_\mu = |\mu\rangle \otimes K_\mu$ is selective \mathcal{PT} -covariant with respect to $(\mathcal{P}, \mathcal{T})$ and $(\mathcal{P}_0 \otimes \mathcal{P}, \mathcal{T}_0 \otimes \mathcal{T})$, that is, $\tilde{K}_\mu \mathcal{PT} \rho \mathcal{PT} \tilde{K}_\mu^\dagger = \mathcal{PT} \tilde{K}_\mu \rho \tilde{K}_\mu^\dagger \mathcal{PT}$. Thus $\tilde{\Phi}(\rho) = \sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu$, where $\rho_\mu = K_\mu \rho K_\mu^\dagger / p_\mu$ with $p_\mu = \text{Tr}(K_\mu \rho K_\mu^\dagger)$. As we have proved that Γ_r satisfies the condition (C2), which implies that $\Gamma_r(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu, \mathcal{P}_0 \mathcal{T}_0 \otimes \mathcal{PT}) \leq \Gamma_r(\rho, \mathcal{PT})$. Moreover,

$$\begin{aligned} & \Gamma_r(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu, \mathcal{P}_0 \mathcal{T}_0 \otimes \mathcal{PT}) \\ &= S(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu^{\mathcal{PT}}) - S(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu) \\ &= \sum_\mu p_\mu S(\rho_\mu^{\mathcal{PT}}) - \sum_\mu p_\mu S(\rho_\mu) \\ &= \sum_\mu p_\mu \Gamma_r(\rho_\mu, \mathcal{PT}). \end{aligned} \quad (\text{A7})$$

where the first equality comes from the Proposition 3 and the second equality comes from the following fact:

$$S(\sum_i p_i |i\rangle\langle i| \otimes \rho_i) = \sum_i p_i S(\rho_i) + H(\{p_i\}), \quad (\text{A8})$$

where $\{|i\rangle\}$ is a set of orthonormal pure states and $H(\{p_i\}) = \sum_i -p_i \log p_i$ is the Shannon entropy of the probability distribution $\{p_i\}$. Therefore, Γ_r satisfies the condition (C2'), that is, Γ_r is a proper \mathcal{PT} -asymmetry measure. \square

Lemma 5. *The measure Γ_r is additive, that is*

$$\Gamma_r(\rho \otimes \sigma, \mathcal{PT} \otimes \mathcal{P}'\mathcal{T}') = \Gamma_r(\rho, \mathcal{PT}) + \Gamma_r(\sigma, \mathcal{P}'\mathcal{T}') \quad (\text{A9})$$

Proof. This comes directly from the representation (A3) of Γ_r . \square

Lemma 6. *The relative entropy of \mathcal{PT} -asymmetry is asymptotically continuous, i.e., for $\|\rho - \sigma\|_1 \leq \epsilon \leq 1/e$, then*

$$|\Gamma_r(\rho, \mathcal{PT}) - \Gamma_r(\sigma, \mathcal{PT})| \leq 2\epsilon \log d + 2\eta(\epsilon), \quad (\text{A10})$$

where d is the dimension of the system and $\eta(\epsilon) = -\epsilon \log \epsilon$.

Proof. Due to $\|\rho - \sigma\|_1 \leq \epsilon$, we have $\|\rho^{\mathcal{PT}} - \sigma^{\mathcal{PT}}\|_1 \leq \epsilon$. Then by Fannes' inequality [71] and (A3), we get the desired result. \square

Appendix B: Skew information of \mathcal{PT} -asymmetry

Proposition 7. *Given the self-inverse unitary operator \mathcal{P} and time reversal operator \mathcal{T} , then Γ_s satisfy the conditions (C1), (C2), (C2') and (C3), so it is a proper \mathcal{PT} -asymmetry measure.*

Proof. If $[\rho, \mathcal{PT}] = 0$, then obviously $\Gamma_s(\rho) = 0$. So we only need to prove the inverse direction. Since $\mathcal{PT}\rho^{1/2}\mathcal{PT} = \mathcal{P}U\mathcal{K}\rho^{1/2}\mathcal{K}U^*\mathcal{P} = \mathcal{P}U(\rho^{1/2})^*U^\dagger\mathcal{P}$ is positive and $(\mathcal{PT}\rho^{1/2}\mathcal{PT})^2 = \mathcal{PT}\rho\mathcal{PT}$, then $\mathcal{PT}\rho^{1/2}\mathcal{PT}$ is the square root of the quantum state $\mathcal{PT}\rho\mathcal{PT}$. Besides, $\text{Tr}(\rho^{1/2}\mathcal{PT}\rho^{1/2}\mathcal{PT}) \leq \text{Tr}(|\rho^{1/2}\mathcal{PT}\rho^{1/2}\mathcal{PT}|) = F(\rho, \mathcal{PT}\rho\mathcal{PT})$, where for any two states ρ_1 and ρ_2 , $F(\rho_1, \rho_2) := \text{Tr}(|\rho_1^{1/2}\rho_2^{1/2}|)$. That is, $\Gamma_s(\rho, \mathcal{PT}) \geq 1 - F(\rho, \mathcal{PT}\rho\mathcal{PT})$. As $\Gamma_s(\rho, \mathcal{PT}) = 0$, then $F(\rho, \mathcal{PT}\rho\mathcal{PT}) = 1$ which means $\rho = \mathcal{PT}\rho\mathcal{PT}$. Thus Γ_s satisfy the condition (C1).

Since $\mathcal{PT}\rho^{1/2}\mathcal{PT}$ is the square root of the quantum state $\mathcal{PT}\rho\mathcal{PT}$, then $\Gamma_s(\rho, \mathcal{PT}) = 1 - \text{Tr}(\rho^{1/2}(\mathcal{PT}\rho\mathcal{PT})^{1/2})$. The convexity of Γ_s :

$$\Gamma_s(p\rho + (1-p)\sigma, \mathcal{PT}) \leq p\Gamma_s(\rho, \mathcal{PT}) + (1-p)\Gamma_s(\sigma, \mathcal{PT})$$

comes from the following famous result [72]:

$$\begin{aligned} & \text{Tr}((p\rho_1 + (1-p)\rho_2)^\alpha (p\sigma_1 + (1-p)\sigma_2)^{1-\alpha}) \\ & \geq p \text{Tr}(\rho_1^\alpha \sigma_1^{1-\alpha}) + (1-p) \text{Tr}(\rho_2^\alpha \sigma_2^{1-\alpha}). \end{aligned}$$

where $\rho_1, \rho_2, \sigma_1, \sigma_2$ are quantum states and $p, \alpha \in [0, 1]$.

Besides, for the quantity $D_\alpha(\rho_1, \rho_2) = \text{Tr}(\rho_1^\alpha \rho_2^{1-\alpha})$ with $\alpha \in (0, 1)$, it holds that [73]:

$$D_\alpha(\rho_1, \rho_2) \leq D_\alpha(\Phi(\rho_1), \Phi(\rho_2)) \quad (\text{B1})$$

where Φ is a quantum operation. Take $\alpha = 1/2$, then we will find that Γ_s satisfies the condition (C2).

Finally, similar to proof in Proposition 2, we take a special self-inverse unitary operator $\mathcal{P}_0 = I$ and time reversal operator $\mathcal{T}_0 = *$ with a set of orthonormal pure states $\{|\mu\rangle\}_\mu$, $|\mu\rangle\langle\mu| \in \text{Sym}(\mathcal{P}_0, \mathcal{T}_0)$. For any selective \mathcal{PT} -covariant operation Φ with $K_\mu(\mathcal{PT}(\cdot)\mathcal{PT})K_\mu^\dagger = \mathcal{PT}K_\mu(\cdot)K_\mu^\dagger\mathcal{PT}$ for any μ , it is easy to verify that the quantum operations $\tilde{\Phi}$ with Kraus operators $\tilde{K}_\mu = |\mu\rangle \otimes K_\mu$ is selective \mathcal{PT} -covariant with respect to $(\mathcal{P}, \mathcal{T})$ and $(\mathcal{P}_0 \otimes \mathcal{P}, \mathcal{T}_0 \otimes \mathcal{T})$, thus $\tilde{\Phi}(\rho) = \sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu$, where $\rho_\mu = K_\mu \rho K_\mu^\dagger / p_\mu$ with $p_\mu = \text{Tr}(K_\mu \rho K_\mu^\dagger)$. As we have proved that Γ_r satisfies the condition (C2), which implies that $\Gamma_s(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu, \mathcal{P}_0 \mathcal{T}_0 \otimes \mathcal{PT}) \leq \Gamma_s(\rho, \mathcal{PT})$. Moreover, it is easy to verify that

$$\begin{aligned} & \Gamma_s(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu, \mathcal{P}_0 \mathcal{T}_0 \otimes \mathcal{PT}) \\ &= \sum_\mu p_\mu \Gamma_s(\rho_\mu, \mathcal{PT}). \end{aligned} \quad (\text{B2})$$

Hence, the condition (C2') holds for Γ_s . \square

Appendix C: Geometric measure of \mathcal{PT} asymmetry

Proposition 8. *Given the self-inverse unitary operator \mathcal{P} and time reversal operator \mathcal{T} , then Γ_F satisfy the conditions (C1), (C2), (C2') and (C3), so it is a proper \mathcal{PT} -asymmetry measure.*

Proof. (C1) is obvious, since $F(\rho, \mathcal{PT}\rho\mathcal{PT}) = 1$ iff $\rho = \mathcal{PT}\rho\mathcal{PT}$. The convexity (C3) of Γ_F comes from the joint concavity of fidelity (See [74] and the reference therein). As fidelity is non-decreasing under CPTP maps (See [74] and the reference therein), thus Γ_F satisfies the condition (C2). Moreover, using a similar method as the proof in Γ_r and Γ_s , we can prove the condition (C2'). Take a special self-inverse unitary operator $\mathcal{P}_0 = I$ and a time reversal operator $\mathcal{T}_0 = *$ with a set of orthonormal pure states $\{|\mu\rangle\}_\mu$, $|\mu\rangle\langle\mu| \in \text{Sym}(\mathcal{P}_0, \mathcal{T}_0)$. For any selective \mathcal{PT} -covariant operation Φ with $K_\mu(\mathcal{PT}(\cdot)\mathcal{PT})K_\mu^\dagger = \mathcal{PT}K_\mu(\cdot)K_\mu^\dagger\mathcal{PT}$ for any μ , it is easy to verify that the quantum operations $\tilde{\Phi}$ with Kraus operators $\tilde{K}_\mu = |\mu\rangle \otimes K_\mu$ is selective \mathcal{PT} -covariant with respect to $(\mathcal{P}, \mathcal{T})$ and $(\mathcal{P}_0 \otimes \mathcal{P}, \mathcal{T}_0 \otimes \mathcal{T})$, thus $\tilde{\Phi}(\rho) = \sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu$, where $\rho_\mu = K_\mu \rho K_\mu^\dagger / p_\mu$ with $p_\mu = \text{Tr}(K_\mu \rho K_\mu^\dagger)$. As we have proved that Γ_r satisfies the condition (C2), which implies that $\Gamma_F(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu, \mathcal{P}_0 \mathcal{T}_0 \otimes \mathcal{PT}) \leq \Gamma_F(\rho, \mathcal{PT})$. More-

over, it is easy to verify that

$$\begin{aligned} & \Gamma_F(\sum_\mu p_\mu |\mu\rangle\langle\mu| \otimes \rho_\mu, \mathcal{P}_0 \mathcal{T}_0 \otimes \mathcal{PT}) \\ &= \sum_\mu p_\mu \Gamma_F(\rho_\mu, \mathcal{PT}). \end{aligned} \quad (\text{C1})$$

Hence, the condition (C2') holds for Γ_F . \square

Lemma 9. *The \mathcal{PT} asymmetry measure Γ_F is continuous, that is*

$$|\Gamma_F(\rho, \mathcal{PT}) - \Gamma_F(\sigma, \mathcal{PT})| \leq 2\sqrt{\|\rho - \sigma\|_1}. \quad (\text{C2})$$

Proof.

$$\begin{aligned} & |\Gamma_F(\rho, \mathcal{PT}) - \Gamma_F(\sigma, \mathcal{PT})| \\ &= |F(\rho, \mathcal{PT}\rho\mathcal{PT}) - F(\sigma, \mathcal{PT}\sigma\mathcal{PT})| \\ &\leq |F(\rho, \mathcal{PT}\rho\mathcal{PT}) - F(\rho, \mathcal{PT}\sigma\mathcal{PT})| \\ &\quad + |F(\rho, \mathcal{PT}\sigma\mathcal{PT}) - F(\sigma, \mathcal{PT}\sigma\mathcal{PT})| \\ &\leq \sqrt{1 - F(\mathcal{PT}\rho\mathcal{PT}, \mathcal{PT}\sigma\mathcal{PT})^2} + \sqrt{1 - F(\rho, \sigma)^2} \\ &= 2\sqrt{1 - F(\rho, \sigma)^2} \leq 2\sqrt{2}\sqrt{1 - F(\rho, \sigma)} \\ &\leq 2\sqrt{\|\rho - \sigma\|_1}. \end{aligned}$$

where the second inequality comes from [75] and the last inequality comes from the Fuchs-van de Graaf inequality [76]. \square

Appendix D: Duality of \mathcal{PT} -Asymmetry and Entanglement

Theorem 10. *Given a two-qubit system with the self-inverse unitary operator $\mathcal{P} = \sigma_y \otimes \sigma_y$ and time reversal operator $\mathcal{T} = *$, for pure bipartite states $|\Psi\rangle$ we have*

$$\Gamma_s(\Psi, \mathcal{PT}) + C(\Psi)^2 = 1, \quad (\text{D1})$$

$$\Gamma_F(\Psi, \mathcal{PT}) + C(\Psi) = 1, \quad (\text{D2})$$

and

$$\Gamma_r(\Psi, \mathcal{PT}) = H\left(\frac{1}{2} - \frac{1}{2}C(\Psi)\right) \quad (\text{D3})$$

where $H(p) = -p\log(p) - (1-p)\log(1-p)$ is Shannon entropy for the probability distribution $\{p, 1-p\}$ and $C(\Psi)$ is the concurrence for pure state Ψ .

For any two-qubit states ρ , the equalities may not hold. However, we still have the following inequality:

$$\Gamma_s(\rho, \mathcal{PT}) + C(\rho)^2 \leq 1, \quad (\text{D4})$$

$$\Gamma_F(\rho, \mathcal{PT}) + C(\rho) \leq 1, \quad (\text{D5})$$

$$\Gamma_r(\rho, \mathcal{PT}) \leq H\left(\frac{1}{2} - \frac{1}{2}C(\rho)\right), \quad (\text{D6})$$

where $C(\rho) = \min \sum_k p_k C(\Psi_k)$ and the minimum is taken over all the pure states decomposition of $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ [49, 50].

In fact,

$$\Gamma_F(\rho, \mathcal{PT}) + \text{CoA}(\rho) = 1 \quad (\text{D7})$$

where the concurrence of assistance $\text{CoA}(\rho) = \max \sum_k p_k C(\Psi_k)$ and the maximum is taken over all the pure states decomposition of $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$ [65, 66].

Proof. The equations (D1), (D2), (D3) come directly from (6), (8) and (10) when $\mathcal{P} = \sigma_y \otimes \sigma_y$ and $\mathcal{T} = *$. Thus we only need to verify that $\mathcal{P} = \sigma_y \otimes \sigma_y$ and $\mathcal{T} = *$ satisfy these three condition. Obviously, $\sigma_y \otimes \sigma_y = (\sigma_y \otimes \sigma_y)^\dagger$ and $(\sigma_y \otimes \sigma_y)^2 = I = \mathcal{T}^2$. Furthermore, as $\mathcal{K}\sigma_y\mathcal{K} = \sigma_y^* = -\sigma_y$, then $\mathcal{K}(\sigma_y \otimes \sigma_y)\mathcal{K} = \sigma_y \otimes \sigma_y$.

Due to the convexity of Γ_s and $C(\rho)$, thus for any pure state decomposition of $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$, we have $\Gamma_s(\rho, \mathcal{PT}) \leq \sum_k p_k \Gamma_s(\Psi_k, \mathcal{PT})$ and $C(\rho) \leq \sum_k p_k C(\Psi_k)$. Based on the convexity of $f(x) = x^2$, we have $C(\rho)^2 \leq \sum_k p_k C(\Psi_k)^2$. Therefore, due to the equality (D1), we get the inequality (D4).

Similarly, (D5) comes directly from the convexity of Γ_r and $C(\rho)$. And (D6) also comes from the convexity of Γ_r , $C(\rho)$ and the concavity of Shannon entropy H , as for any pure states decomposition of $\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|$,

$$\begin{aligned} \Gamma_r(\rho, \mathcal{PT}) &\leq \sum_k p_k \Gamma_r(\Psi_k, \mathcal{PT}) \\ &= \sum_k p_k H\left(\frac{1}{2} - \frac{1}{2} C(\Psi_k)\right) \\ &\leq H\left(\frac{1}{2} - \frac{1}{2} \sum_k p_k C(\Psi_k)\right) \\ &\leq H\left(\frac{1}{2} - \frac{1}{2} C(\rho)\right). \end{aligned}$$

Finally, (D7) comes directly from the definition of concurrence of assistance (CoA) and Eq.5. \square

Appendix E: \mathcal{PT} -symmetric dynamics

Consider the unitary operator V which is \mathcal{PT} -covariant, that is

$$\mathcal{PT}V\rho V^\dagger \mathcal{PT} = V\mathcal{PT}\rho \mathcal{PT}V^\dagger, \text{ for any } \rho, \quad (\text{E1})$$

then

$$V\mathcal{PT} = e^{i\theta} \mathcal{PT}V. \quad (\text{E2})$$

Besides, the unitary operator V is called \mathcal{PT} -invariant unitary if $[V, \mathcal{PT}] = 0$. If $[V, \mathcal{PT}] = 0$, then $[V^\dagger, \mathcal{PT}] = 0$. Note that, if the Hamiltonian H satisfying $\{H, \mathcal{PT}\} = 0$ where $\{A, B\} = AB + BA$, then the unitaries e^{iHt} satisfy $[e^{iHt}, \mathcal{PT}] = 0$.

Similar to entanglement [77] and coherence [78], we can also consider the state transformation under \mathcal{PT} -covariant operation.

Proposition 11. Pure state $|\psi\rangle$ can be transformed to $|\phi\rangle$ under selective \mathcal{PT} -covariant operations if and only if $|\langle\psi|\mathcal{PT}|\psi\rangle| \leq |\langle\phi|\mathcal{PT}|\phi\rangle|$.

Proof. The "only if" part is obvious, we only need to prove the "if" part. Define $\mathcal{H}_{\mathcal{PT}} = \{|\psi\rangle \in \mathcal{H} : \mathcal{PT}|\psi\rangle = |\psi\rangle\}$, then $\mathcal{H}_{\mathcal{PT}}$ is a real Hilbert space with $\dim \mathcal{H}_{\mathcal{PT}} = \dim \mathcal{H}$ [50, 60]. Thus the basis of $\mathcal{H}_{\mathcal{PT}}$ is also a basis of \mathcal{H} , which is called \mathcal{PT} -invariant basis. Here we use $\{|\psi\rangle_O\}_{i=1}^d$ and $\{|\psi\rangle_N\}_{i=1}^d$ to denote the initial given basis of \mathcal{H} and the \mathcal{PT} -invariant basis, $|\psi\rangle_O$ and $|\psi\rangle_N$ to denote the representation of state $|\psi\rangle$ in the basis $\{|\psi\rangle_O\}_{i=1}^d$ and $\{|\psi\rangle_N\}_{i=1}^d$, respectively. Obviously, there exists a unitary operator U such that $|\psi\rangle_N = U|\psi\rangle_O$ for any ψ . Besides, the \mathcal{PT} operator in the new basis $\{|\psi\rangle_N\}_{i=1}^d$ is equivalent to the complex conjugation with respect to this basis, denoted by \mathcal{K}_N . Then $\mathcal{PT}|\psi\rangle_O = \mathcal{K}_N|\psi\rangle_N$ and $\langle\psi|\mathcal{PT}|\psi\rangle \equiv \langle\psi|\mathcal{PT}|\psi\rangle_O = \langle\psi|\mathcal{K}_N|\psi\rangle_N = \langle\psi|U^\dagger \mathcal{K}_N U|\psi\rangle_O$, which implies that $\mathcal{PT} = U^\dagger \mathcal{K}_N U$. Thus $|\langle\psi|\mathcal{PT}|\psi\rangle| \leq |\langle\phi|\mathcal{PT}|\phi\rangle|$ is equivalent to $|\langle\psi|\mathcal{K}_N|\psi\rangle_N| \leq |\langle\phi|\mathcal{K}_N|\phi\rangle_N|$. Due to Ref.[42], there exists a CPTP map Φ with Kraus operators $\{K_\mu\}$ such that $[K_\mu, \mathcal{K}_N] = 0$ and $\Phi(|\psi\rangle\langle\psi|_N) = |\phi\rangle\langle\phi|_N$. Therefore, in the initial given basis $\{|\psi\rangle_O\}_{i=1}^d$, the quantum operation $\tilde{\Phi}$ with $\tilde{K}_\mu = U^\dagger K_\mu U$ satisfy the conditions $[\tilde{K}_\mu, \mathcal{PT}] = 0$ and $\tilde{\Phi}(|\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi|$. \square

The \mathcal{PT} -asymmetry measure introduced here is connected with the time reversal symmetry monotone in [42] up to a unitary. However, it is hard to find the unitary U such that $\Gamma(\rho, \mathcal{PT}) = \Gamma(U\rho U^\dagger, \mathcal{K}_N)$ where \mathcal{K}_N is the complex conjugation with the \mathcal{PT} -invariant basis. And such a unitary, especially a global unitary on the composite system, may ruin the nonlocality of \mathcal{PT} -symmetry.

Except the \mathcal{PT} -covariant operations, we can also consider the operations which map the \mathcal{PT} -symmetric state to \mathcal{PT} -symmetric state, that is $\Phi(\text{Sym}(\mathcal{P}, \mathcal{T})) \subset \text{Sym}(\mathcal{P}, \mathcal{T})$. Such operations are called \mathcal{PT} -preserving operations. Obviously, all \mathcal{PT} -covariant operations are \mathcal{PT} -preserving operations. Moreover, for any operations $\Phi = \sum_\mu K_\mu(\cdot)K_\mu^\dagger$ with $K_\mu(\text{Sym}(\mathcal{P}, \mathcal{T}))K_\mu^\dagger \subset \text{Sym}(\mathcal{P}, \mathcal{T})$, we call such operation selective \mathcal{PT} -preserving operations, which is similar to the definition of incoherent operations [15]. We can weaken the conditions (C2) and (C2') to the following conditions:

(C2a) Monotone under \mathcal{PT} -preserving operations $\Phi_{\mathcal{PT}_{pre}}$, that is $\Gamma(\Phi_{\mathcal{PT}_{pre}}(\rho), \mathcal{PT}) \leq \Gamma(\rho, \mathcal{PT})$.

(C2'a) Monotone under selective \mathcal{PT} -preserving operations, that is $\sum_\mu p_\mu \Gamma(\rho_\mu, \mathcal{PT}) \leq \Gamma(\rho, \mathcal{PT})$, where $K_\mu \text{Sym}(\mathcal{P}, \mathcal{T}) K_\mu^\dagger \subset \text{Sym}(\mathcal{P}, \mathcal{T})$ and $\rho_\mu = K_\mu \rho K_\mu^\dagger / p_\mu$ with $p_\mu = \text{Tr}(K_\mu \rho K_\mu^\dagger)$.

Therefore, we can also consider the \mathcal{PT} -asymmetry monotone which satisfy the conditions (C1), (C2a), (C2'a) and (C3), which will be explored in a future work.